Exact solitonic solutions for optical media with $\chi^{(2)}$ nonlinearity and PT-symmetric potentials

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Abstract. We study the $\chi^{(2)}$ system of equations with the PT-symmetric potentials. We obtain exact solitary wave solutions for this system for some forms of the PT-symmetric potentials.

1. Introduction

Recently a great attention has been devoted to the investigation of quantum and classical systems with PT-symmetric non-Hermitian Hamiltonians. It was shown in the seminal paper [1] that such type of Hamiltonians can have real eigenvalues. This result has growing applications in optics, Bose-Einstein condensates, plasmonics etc. In particular in optics due to the analogy between the Schrodinger equation and the paraxial wave equation for optical beams the PT-symmetry imposes the condition on the complex refractive index $n(x) = n_r(x) + in_i(x)$: even in space of a real part of refractive index $n_r(x) = n_r(-x)$ and odd in space imaginary part $n_i(x) = -n(-x)$. Recently effects of PT- symmetry have been observed in optical experiments[2]. The nonlinearity leads to new effects in the PT- symmetric systems, such as the nonreciprocity[3, 4], smoothing the spectral singularity in transmission [5] etc. It was shown recently that in nonlinear wave equations with PT- symmetric extension solitonic solutions can exist [6, 7, 8, 9, 10]. Particular interest represents search of exact solutions for the nonlinear waves in media with PT-symmetric modulations of parameters. In the case of the media with the Kerr nonlinearity described by the nonlinear Schrödinger equation with inhomogeneous in space loss/gain parameters such solutions were found for linear PT-potentials in the works [6, 7, 11, 12, 13, 14], for nonlinear PT-potentials - in the work[7, 8]. Recent numerical simulations of the $\chi^{(2)}$ system with PT - symmetric potential have showed the existence of stable bright solitons[15]. In this work we will study the $\chi^{(2)}$ system with PT - symmetric potentials, describing wave processes in quadratically nonlinear media and we will find the exact solitary solutions describing localized wave packets in such systems.

2. The model

Let us consider the $\chi^{(2)}$ system describing the FH and SH propagation in quadratically nonlinear media with PT-symmetric potentials[16]

$$iu_z + d_1 u_{xx} + V_1(x)u + iW_1(x)u = u^* v, \quad (1)$$
$$iv_z + d_2 v_{xx} + qv + V_2(x)v + iW_2(x)v = u^2, \quad (2)$$

with $d_{1,2}$ are diffraction coefficients, q is the phase missmatch. Here $V_{1,2}(x)$ are real parts of the refraction index. PT-symmetry impose restriction on $V_{1,2}$ to be even functions of x and and on the imaginary potential $W_{1,2}(x)$ describing the inhomogeneous in space gain/loss, to be odd functions of x. Introducing representation

$$u(x,z) = U(x)e^{-i\omega z}, v(x,z) = V(x)e^{-2i\omega z},$$

we obtain the system of equations for $U(x)$ and $W(x)$

$$\omega U + d_1 U_{xx} + V_1(x)U + iW_1(x)U = U^*V, \tag{3}$$
$$\sigma V + d_2 V_{xx} + V_2(x)V + iW_2(x)V = U^2, \tag{4}$$

where $\sigma = 2\omega + q$. It is useful to introduce the amplitude-phase variables by

$$U(x) = \rho_1(x)e^{i\theta(x)},\ V(x) = \rho_2(x)e^{2i\theta(x)}.$$

Then we obtain the system:

$$\omega\rho_1 + d_1\rho_{1,xx} - d_1\rho_1(\theta_x)^2 + V_1(x)\rho_1 = \rho_1\rho_2, \tag{5}$$
$$\sigma\rho_2 + d_2\rho_{2,xx} - 4d_2\rho_2(\theta_x)^2 + V_2(x)\rho_2 = \rho_1^2, \tag{6}$$
$$W_1(x)\rho_1^2 = -d_1(\rho_1^2\theta_x)_x, \tag{7}$$
$$W_2(x)\rho_2^2 = -2d_2(\rho_2^2\theta_x)_x. \tag{8}$$

To find the exact solutions of the system (5) we will apply the inverse engineering approach i.e. construct the potentials admitting the exact solutions[17, 7].

3. Bright soliton solutions

First we consider the case of localized solutions in the form of bright-bright solitonic solutions.

1. Let look for solutions of the form:

$$\rho_1 = A\mathrm{sech}^2(rx),\ \rho_2 = B\mathrm{sech}^2(rx). \tag{9}$$

This form, for constant parameters case, has been introduced by Karamzin-Sukhorukov[18]. For the phase we impose the form:

$$\theta_x(x) = C\mathrm{sech}(rx). \tag{10}$$

Then we obtain from (5) for the $W_1(x)$

$$W_1(x) = 5Cd_1 r\mathrm{sech}(rx)\tanh(rx) = W_{01}\mathrm{sech}(rx)\tanh(rx). \tag{11}$$

Analogously we have found

$$W_2(x) = 10Cd_2 r\mathrm{sech}(rx)\tanh(rx) = W_{02}\mathrm{sech}(rx)\tanh(rx).$$

Then

$$C = \frac{W_{01}}{5d_1 r} = \frac{W_{02}}{10d_2 r}.$$

We obtain the relation between W_{01}, W_{02}

$$W_{02} = 2\frac{d_2}{d_1}W_{01}.$$

From equations for amplitudes we have

$$V_i = V_{0i}\mathrm{sech}^2(rx),\ \omega = -4d_1r^2, \sigma = -4d_2r^2, q = 4r^2(2d_1 - d_2). \tag{12}$$

$$B = (V_{01} - 6d_1r^2 - \frac{W_{01}^2}{25r^2d_1}),\ A^2 = B(V_{02} - 6d_2r^2 - \frac{4d_2W_{01}^2}{25r^2d_1^2}). \tag{13}$$

The phase is:

$$\theta(x) = \frac{2W_{01}}{5r^2d_1}\mathrm{arctg}(\tanh(rx/2)). \tag{14}$$

For example if $r = 1, d_1 = 1/2, d_2 = 1/4$ we obtain that the phase mismatch should be $q = 3$. Taking into account the condition $A^2 > 0$, dynamical breaking of the PT- symmetry occurs, when $B > 0$ and the strength of imaginary part of potential excess the value

$$W_{1c}(x) = 5r\sqrt{d_1(V_{01} - 6d_1r^2)}, \tag{15}$$

or when $B < 0$ and

$$W_{1c}(x) < 5r\sqrt{d_1(V_{01} - 6d_1r^2)}, \tag{16}$$

with $V_{01} = 2V_{02}, d_1 = 2d_2$. Existence and stability of the solution when the conditions above are not fulfilled requires a separate consideration.

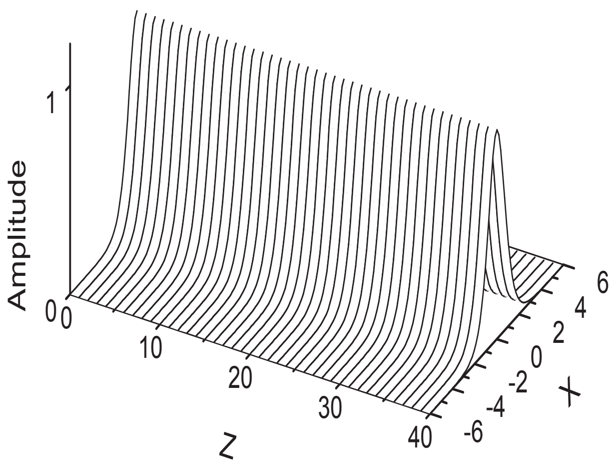

Figure 1. Evolution of the amplitude $|u(x, z)|$ with initial conditions in form of exact solution (9-10). Parameters: $A = 1$, $B = 1$, $r = 1$, $d_1 = 0.5$, $d_2 = 0.25$, $W_{01} = 1$.

2. Second choice of the solution is:

$$\rho_1(x) = A\mathrm{sech}(rx),\ \rho_2 = B\mathrm{sech}^2(rx). \tag{17}$$

This form for constant parameters of $\chi^{(2)}$ system has been introduced in [19]. Again we will assume the functional dependence for the phase gradients in the form (10). Then we obtain for $W_i(x)$ relations:

$$\begin{aligned} W_1 &= 3Cd_1r\mathrm{sech}(rx)\tanh(rx) = W_{01}\mathrm{sech}(rx)\tanh(rx), \\ W_2 &= 10Cd_2r\mathrm{sech}(rx)\tanh(rx) = W_{02}\mathrm{sech}(rx)\tanh(rx), \\ C &= \frac{W_{01}}{3rd_1} = \frac{W_{02}}{10d_2r}. \end{aligned} \tag{18}$$

Then we have

$$W_{02} = \frac{10d_2W_{01}}{3d_1}.$$

It can be shown, that

$$\omega = -d_1r^2.$$

For soliton amplitudes we find respectively:

$$B = V_{01} - 2d_1r^2 - \frac{W_{01}^2}{9r^2d_1}, A^2 = B(\sigma + 4d_2r^2). \tag{19}$$

For real potentials we have expressions

$$V_1(x) = V_{01}\mathrm{sech}^2(rx), V_2(x) = V_{02}\mathrm{sech}^2(rx,), \tag{20}$$

where

$$V_{02} = 6d_2r^2(1 + \frac{2W_{01}^2}{27d_1^2r^4}). \tag{21}$$

Again we have the dynamical breaking of the PT symmetry above threshold value for the strength of the imaginary part of potential W

$$W_{1c} = 3r\sqrt{d_1(V_{01} - 2d_1r^2)}.$$

It is useful to consider the large negative mismatch case when $|q| >> 1$. Then $v \approx -u^2/|q|$ and the system is reduced to the PT-symmetric extension of the single NLS equation(the so-called a cascading limit case) The solitonic solution for the potentials $V = V_0\mathrm{sech}^2(x), W = W_0\mathrm{sech}(x)\tanh(x)$ is well known[6] and coincides with the solution (19), when $|q| >> 1, r = 1, d_1 = 1$.

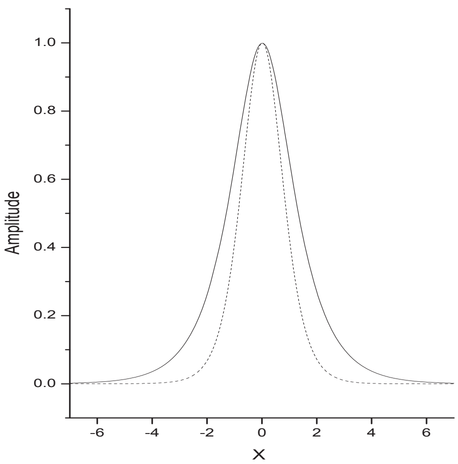

Figure 2. Amplitude of the exact solution (16) $|u(x)|$ solid line and $|v(x)|$ dashed line at $z = 0$. Parameters: $A = 1$, $B = 1$, $r = 1$, $d_1 = 0.5$, $d_2 = 0.25$, $W_{01} = 1$.

3.Third choice of solutions is:

$$\rho_1(x) = A\mathrm{sech}(rx)\tanh(rx)e^{i\theta(x)+i\pi/2},\ \ \rho_2(x) = B\mathrm{sech}^2(rx)e^{2i\theta(x)}. \tag{22}$$

This type of solitonic solutions for an homogeneous $\chi^{(2)}$ media is considered in [20, 21]. Let take the phase of the form

$$\theta_x = C\mathrm{sech}(rx)\tanh(rx).$$

Looking for the imaginary part of potential of the form

$$W_1(x) = W_1(\mathrm{sech}(rx) - 2\mathrm{sech}^3(rx)), W_2(x) = W_2\mathrm{sech(rx)}(5 - 6\mathrm{sech}^2(rx))$$

we obtain

$$C = \frac{W_1}{3d_1 r} = \frac{W_2}{2d_2 r} = \sqrt{\frac{|V_1^{(2)}|}{d_1}} = \sqrt{\frac{|V_2^{(2)}|}{4d_2}}.$$

Also we find

$$\begin{aligned} A^2 &= -(\sigma + 4d_2 r^2)B, B = (-(V_1^{(1)} + V_1^{(2)}) + 6d_1 r^2), \\ V_i(x) &= V_i^{(1)}\mathrm{sech}^2(x) + V_i^{(2)}\mathrm{sech}^4(x), i = 1, 2; \theta(x) = -\frac{W_1}{3d_1 r^2}\mathrm{sech}(rx),\ \omega = -d_1 r^2, \\ \sigma &= 2d_2 r^2 - (V_2^{(1)} + V_2^{(2)}). \end{aligned} \tag{23}$$

So, as one can note, this is the example of exact solution in the complex non PT symmetric potential, since both real and imaginary parts of the potential are even functions of the x variable.

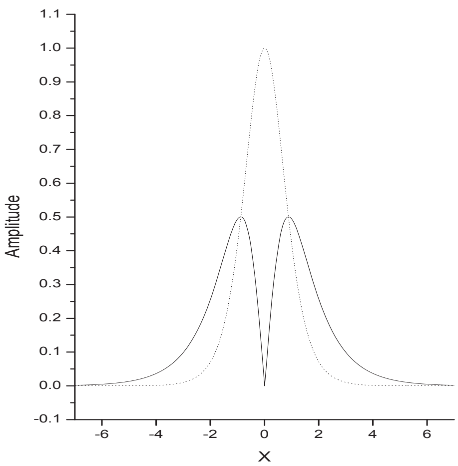

Figure 3. Amplitude of the exact solution (21) $|u(x)|$ solid line and $|v(x)|$ dashed line at $z = 0$. Parameters: $A = 1$, $B = 1$, $r = 1$, $d_1 = 0.5$, $d_2 = 0.25$, $W_{01} = 1$.

4. Conclusion

. We have found exact solitary solutions for quadratically nonlinear media with PT- symmetric potentials. The solutions were in the form of bright-bright solitons. Also the example of exact solution in non PT symmetric potential is given. The investigation of stability of obtained exact solutions is important problem and requires a separate analysis.

Acknowledgments

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